A point P in the plane of triangle ABC

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Find a point P in the plane of a given triangle ABC, such that

$|AP|^2/b^2 + |BP|^2/c^2 + |CP|^2/a^2$ is minimal,

where $a = BC, b = CA$ and $c = AB$.

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Let $\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ be some vectors which represent points P,A,B,C on the plane, respectively.

Then $a = \|\mathbf{b} - \mathbf{c}\|, b = \|\mathbf{c} - \mathbf{a}\|, c = \|\mathbf{a} - \mathbf{b}\|$ and our problem is:

Find a vector $\mathbf{x}$ which minimize $F(\mathbf{x}) := \dfrac{1}{b^2}\|\mathbf{x} - \mathbf{a}\|^2 + \dfrac{1}{c^2}\|\mathbf{x} - \mathbf{b}\|^2 + \dfrac{1}{a^2}\|\mathbf{x} - \mathbf{c}\|^2$.

This problem is particular case of more general problem:

Find a vector $\mathbf{x}$ which minimize $F(\mathbf{x}) := p\|\mathbf{x} - \mathbf{a}\|^2 + q\|\mathbf{x} - \mathbf{b}\|^2 + r\|\mathbf{x} - \mathbf{c}\|^2$ for any given positive weights p, q, r.

Asssming for convenience $p + q + r = 1$ and denoting $\mathbf{m} := p\mathbf{a} + q\mathbf{b} + r\mathbf{c}$ we obtain

$F(\mathbf{x}) = p((\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})) + q((\mathbf{x} - \mathbf{b}) \cdot (\mathbf{x} - \mathbf{b})) + r((\mathbf{x} - \mathbf{c}) \cdot (\mathbf{x} - \mathbf{c})) =$

$(\mathbf{x} \cdot \mathbf{x}) - 2(\mathbf{m} \cdot \mathbf{x}) + p(\mathbf{a} \cdot \mathbf{a}) + q(\mathbf{b} \cdot \mathbf{b}) + r(\mathbf{c} \cdot \mathbf{c}) =$

$(\mathbf{x} - \mathbf{m}) \cdot (\mathbf{x} - \mathbf{m}) + p(\mathbf{a} \cdot \mathbf{a}) + q(\mathbf{b} \cdot \mathbf{b}) + r(\mathbf{c} \cdot \mathbf{c}) - (\mathbf{m} \cdot \mathbf{m}) \geq \sum p(\mathbf{a} \cdot \mathbf{a}) - (\mathbf{m} \cdot \mathbf{m}) =$

$\sum p(\mathbf{a} \cdot \mathbf{a}) - \sum p^2(\mathbf{a} \cdot \mathbf{a}) - \sum pq(\mathbf{a} \cdot \mathbf{b}) - \sum pr(\mathbf{a} \cdot \mathbf{c}) = \sum pq(\mathbf{a} \cdot \mathbf{a}) + \sum pr(\mathbf{a} \cdot \mathbf{a}) -$

$\sum pq(\mathbf{a} \cdot \mathbf{b}) - \sum pr(\mathbf{a} \cdot \mathbf{c}) = \sum pq(\mathbf{a} \cdot \mathbf{a}) + \sum qp(\mathbf{b} \cdot \mathbf{b}) - 2\sum pq(\mathbf{a} \cdot \mathbf{b}) =$

$\sum pq((\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})) = pqc^2 + qra^2 + rpb^2 = F(\mathbf{m})$.

Thus, $\min F(\mathbf{x}) = F(\mathbf{m}) = pqc^2 + qra^2 + rpb^2$ and for weights without normalization

$\min F(\mathbf{x}) = F\left(\dfrac{p\mathbf{a} + q\mathbf{b} + r\mathbf{c}}{p + q + r}\right) = \dfrac{pqc^2 + qra^2 + rpb^2}{p + q + r}$.

In particular for $(p, q, r) = \left(\dfrac{1}{b^2}, \dfrac{1}{c^2}, \dfrac{1}{a^2}\right)$ we obtain $pqc^2 + qra^2 + rpb^2 =$

$\dfrac{1}{b^2c^2} \cdot c^2 + \dfrac{1}{c^2a^2} \cdot a^2 + \dfrac{1}{a^2b^2} \cdot b^2 = \dfrac{1}{b^2} + \dfrac{1}{a^2} + \dfrac{1}{c^2}$. Thus, $\min F(\mathbf{x}) = 1$

and attained in unique point P represnted by vector $\mathbf{x}_* = \dfrac{b^{-2}\mathbf{a} + c^{-2}\mathbf{b} + a^{-2}\mathbf{c}}{b^{-2} + c^{-2} + a^{-2}} =$

$\dfrac{c^2a^2\mathbf{a} + a^2b^2\mathbf{b} + b^2c^2\mathbf{c}}{c^2a^2 + a^2b^2 + b^2c^2}$, that is P is detrmined by barycentric coordinates $c^2a^2 \div a^2b^2 \div b^2c^2$

with respect to triangle ABC.